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A Note on a Very Simple Property about the Volume of a *n*-Simplex and the Centroids of its Faces

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ABSTRACT

A n–simplex is the convex hull of a set of $(n + 1)$ independent points affine to a n–dimensional Euclidean space. A 0–simplex would be a point, a segment would be a 1–simplex, a 2–simplex would be a triangle and a 3–simplex a tetrahedron. If we connect to each other the centroids of all the $(n - 1)$ faces of a n–simplex, we will obtain another n-simplex (dual n-simplex). The volume ratio between a n-simplex and its dual is just n^{-n}.

KEYWORDS: *n–simplex, centroid.*

RESUMEN

Un n–simplex es la envoltura convexa de un conjunto de $(n + 1)$ puntos independientes afines en un espacio euclídeo de dimensión n. Así un 0–simplex sería un punto, un 1–simplex sería un segmento, un 2–simplex sería un triángulo y un 3–simplex sería un tetraedro. Si unimos entre sí los centroides o baricentros de las $(n - 1)$ caras de un n–simplex, se obtiene otro n–simplex (n–simplex dual). Demostraremos que la razón de volúmenes entre un n–simplex y su dual es exactamente n^{-n}.

PALABRAS CLAVE: *n–simplex*, *baricentro.*

INTRODUCTION

There is a well-known theorem [1] that states that if we join the middle points of the sides of a triangle $\Delta V_1V_2V_3$ (see Fig. 1), we will obtain another triangle $\Delta M_1M_2M_3$, which is similar to the former

$$\Delta V_1V_2V_3 \sim \Delta M_1M_2M_3, \tag{1}$$

and whose similarity ratio is 1 : 2,

$$V_iV_j = 2M_iM_j, \qquad i, j = 1, 2, 3. \tag{2}$$

Equations (1) and (2) can be proved easily applying Thales' theorem, realizing that the following triangles are similar,

$$\Delta V_1V_2V_3 \sim \Delta M_iM_jM_k,$$

where the indices (i, j, k) are the cyclic permutations of (1, 2, 3). According to (1) and (2) and applying Galileo's square-cube law [3, Prop. VIII], the area ratio of both triangles is $(1/2)^2 = 1/4$, that is,

$$[V_1V_2V_3] = \frac{1}{4}[M_iM_jM_k]. \tag{3}$$

According to Fig. 1, note that $\Delta M_1M_2M_3$ is rotated π rad with respect to $\Delta V_1V_2V_3$.

If we interpret $\Delta V_1V_2V_3$ as a 2–simplex and the middle points M_i $(i = 1, 2, 3)$ as the side centroids, we can generalize the result given in (3) to a higher number of dimensions

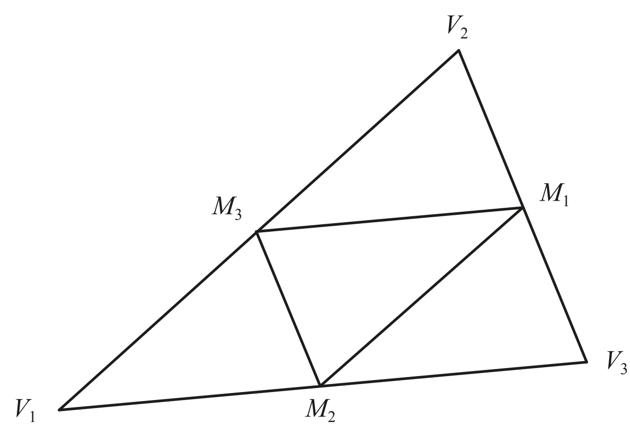

Figura 1. The medial triangle is just one fourth of the area of the triangle in which it is inscribed.

For this purpose, let us connect the centroids of all the $(n - 1)$–faces of a n–simplex to each other, in order to obtain another n–simplex (dual n–simplex). We can establish the following conjecture.

Conjecture 1. The volume ratio between a n–simplex and its dual is just n^{-n}.

Equation (3) proves this conjecture for $n = 2$. Section 2 proves this conjecture for $n = 3$ and Section 3 for $n \geq 2$, $n \in \mathbb{N}$.

THE CENTROID TRIANGLE

Lemma 2. Let us consider a plane polygon V of vertices $V_1, V_2, \ldots, V_n$ and a point P in the space. The point P defines with each side of the polygon n sub–triangles (see Fig. 2). Joining the centroids of each sub-triangle, we get a polygon V' of vertices $V_1', V_2', \ldots, V_n'$ which is always of the same shape and area, regardless the point P chosen.

Proof. Consider three consecutive vertices of the polygon V_1, V_2 and V_3 as it is shown in Fig. 2. The centroids of the 2 sub–triangles given by these three vertices and P are V_1' and V_2'. Since the polygon V lies on a plane, we can consider that the Cartesian coordinates of the vertices V_1, V_2, V_3 are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$ respectively. If the Cartesian coordinates of P are (x_P, y_P, z_P), then

$$V_1' = \frac{1}{3}(x_1 + x_2 + x_P, \mathrm{y}_1 + y_2 + y_P, \mathrm{z}_P)$$

$$V_2' = \frac{1}{3}(x_2 + x_3 + x_P, \mathrm{y}_2 + y_3 + y_P, \mathrm{z}_P)$$

Consider now the vector

$$\overrightarrow{V_1'V_2'} = \frac{1}{3}(x_3 - x_1, y_3 - y_1, 0)$$

Since $\overrightarrow{V_1'V_2'}$ does not depend on the coordinates of P, then, regardless the point P chosen, all the possible segments $V_1'V_2'$ are always parallel one to each other and of the same size. Therefore, all the V' polygons constructed by joining all the centroids of the sub–triangles must be of the same shape and area. Moreover, the new polygon V' lies on a parallel plane at a distance $z_P/3$ to the plane that contains V.

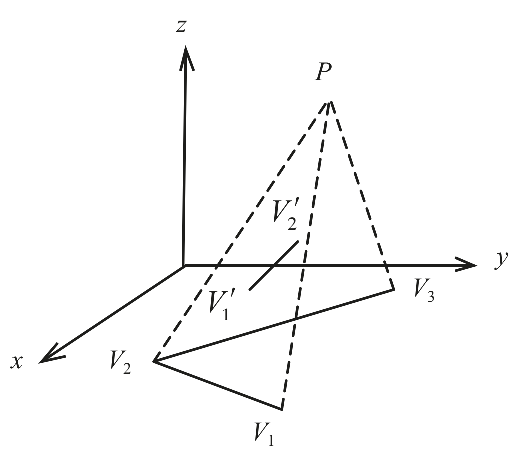

Figura 2. Centroids V_1' and V_2' of two sub-triangles of a polygon.

Lemma 3. Be a plane triangle of vertices V_1, V_2 and V_3 and a point P in the space. The point P defines with each side of $\Delta V_1V_2V_3$ another 3 sub–triangles. Joining the centroids of each sub-triangle, we always get another $\Delta V_1'V_2'V_3'$ (centroid triangle) similar to $\Delta V_1V_2V_3$ and of $1/9$ of its area, regardless the point P chosen.

Proof. By using lemma 2, every $\Delta V_1'V_2'V_3'$ are of the same shape and size, regardless the point P chosen. Let us choose as point P the centroid G of $\Delta V_1V_2V_3$. In Fig. 3 the centroids of the 3 sub–triangles are given by G_1, G_2 and G_3, and are located on the medians of $\Delta V_1V_2V_3$. Remembering that the centroid of a triangle divides the median in the ratio 2 : 1 then,

$$GV_i = 2GM_i = 3GG_i, \qquad i = 1, 2, 3, \tag{4}$$

thus

$$\Delta G_iGG_j \sim \Delta V_iGV_j,$$

and the sides of $\Delta G_1G_2G_3$ are parallel to the sides of $\Delta V_1V_2V_3$. Therefore

$$\Delta G_1G_2G_3 \sim \Delta V_1V_2V_3,$$

being the similarity ratio for these triangles 1 : 3,

$$V_iV_j = 3G_iG_j, \qquad i \neq j, \quad i, j = 1, 2, 3. \tag{5}$$

Applying Galileo's square-cube law, the area ratio between $\Delta V_1V_2V_3$ and $\Delta G_1G_2G_3$ is 1 : 9, that is,

$$[V_1V_2V_3] = \frac{1}{9}[G_1G_2G_3].$$

Remark 4. Note that $\Delta G_1G_2G_3$ is rotated π rad with respect to $\Delta V_1V_2V_3$.

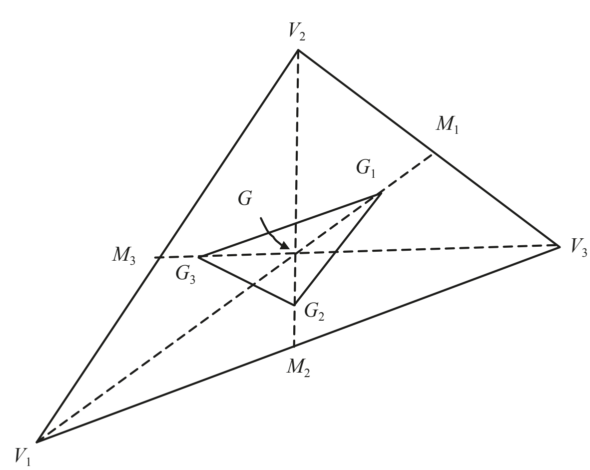

Figura 3. Centroids G_i of the sub–triangles taking P as the centroid G of $\Delta V_1V_2V_3$.

Theorem 5. Connecting the centroids of each face of a tetrahedron (see Fig. 4), we obtain another tetrahedron (dual tetrahedron) similar to the first one and of $1/27$ of its volume.

Proof. Let us use lemma 3 taking as $\Delta V_1V_2V_3$. a face of the tetrahedron and as point P the opposite vertex. Then, all the faces of the dual tetrahedron are similar and parallel to the opposite faces of the tetrahedron (but rotated π rad), so both tetrahedrons are similar. Moreover, by the similarity ratio between the sides of both tetrahedrons is 1/3. Therefore, applying Galileo's square–cube law, their volume ratio is $(1/3)^3 = 1/27$.

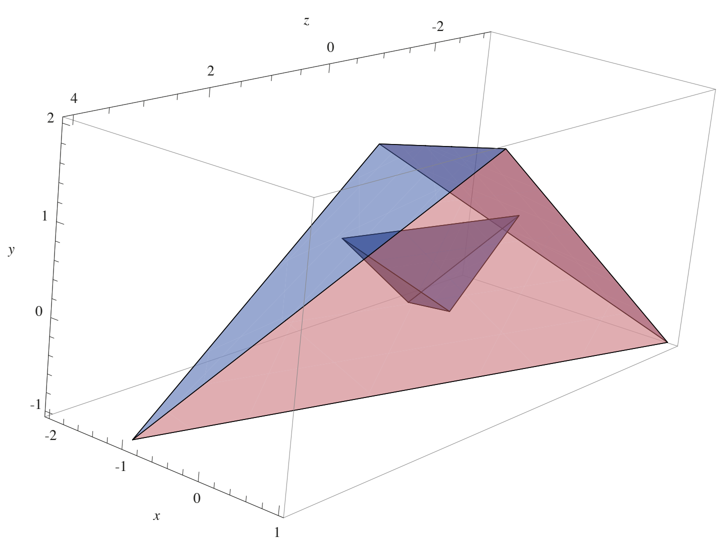

Figura 4. Example of a tetrahedron and its dual.

GENERALIZATION

The previous theorem proves conjecture 1 for $n = 3$. Now, let us prove the conjecture for an arbitrary number n of dimensions.

Theorem 6. Let us consider a n–simplex V whose vertices are given by $V_1, V_2, \dots, V_{n+1}$. Connecting the centroids of the $(n-1)$–faces of V each other, we get another n–simplex V'. The volume ratio of V' with respect to V is n^{-n}.

Proof. The Cartesian coordinates of each vertex of V are

$$V_i = (x_{1i}, x_{2i}, \dots, x_{ni}), \quad i = 1, \dots, n+1,$$

and of V' are

$$V_i' = (x_{1i}', x_{2i}', \dots, x_{ni}') = \frac{1}{n}\sum_{j \neq i}^{n+1} (x_{1j}, x_{2j}, \dots, x_{nj}), \quad i = 1, \dots, n+1. \tag{6}$$

By using Lagrange's formula [2], the volume of $\mathcal{V}$ is given by

$$\mathcal{V}=\frac{1}{n!}\begin{Vmatrix} x_{11} & \cdots & x_{n1} & 1 \\ x_{12} & \cdots & x_{n2} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{1,n+1} & \cdots & x_{n,n+1} & 1 \end{Vmatrix} \tag{7}$$

where $\|A\|$ denotes the absolute value[1] of the determinant of *A*. Then, taking into account (6), the volume of $\mathcal{V}'$ is given by

$$\mathcal{V}'=\frac{1}{n!}\begin{Vmatrix} x'_{11} & \cdots & x'_{n1} & 1 \\ x'_{12} & \cdots & x'_{n2} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x'_{1,n+1} & \cdots & x'_{n,n+1} & 1 \end{Vmatrix}=\frac{1}{n!n^n}\begin{Vmatrix} \sum_{j\neq 1}^{n+1} x_{1j} & \cdots & \sum_{j\neq 1}^{n+1} x_{nj} & 1 \\ \sum_{j\neq 2}^{n+1} x_{1j} & \cdots & \sum_{j\neq 2}^{n+1} x_{nj} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \sum_{j\neq n+1}^{n+1} x_{1j} & \cdots & \sum_{j\neq n+1}^{n+1} x_{nj} & 1 \end{Vmatrix}. \tag{8}$$

Note that we can rewrite (7) and (8) transforming each row r_k into a new row r'_k according to

$$r'_k = r_k - r_{k+1}, \qquad k=1,\ldots,n,$$

thus

$$\mathcal{V}=\frac{1}{n!}\begin{Vmatrix} x_{11}-x_{12} & \cdots & x_{n1}-x_{n2} & 0 \\ x_{12}-x_{13} & \cdots & x_{n2}-x_{n3} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{1n}-x_{1,n+1} & \cdots & x_{nn}-x_{n,n+1} & 0 \\ x_{1,n+1} & \cdots & x_{n,n+1} & 1 \end{Vmatrix}=\frac{1}{n!}\begin{Vmatrix} x_{11}-x_{12} & \cdots & x_{n1}-x_{n2} \\ x_{12}-x_{13} & \cdots & x_{n2}-x_{n3} \\ \vdots & \ddots & \vdots \\ x_{1n}-x_{1,n+1} & \cdots & x_{nn}-x_{n,n+1} \end{Vmatrix}, \tag{9}$$

and, similarly,

$$\mathcal{V}'=\frac{1}{n!n^n}\begin{Vmatrix} x_{12}-x_{11} & \cdots & x_{n2}-x_{n1} \\ x_{13}-x_{12} & \cdots & x_{n3}-x_{n2} \\ \vdots & \ddots & \vdots \\ x_{1,n+1}-x_{1n} & \cdots & x_{n,n+1}-x_{nn} \end{Vmatrix}=\frac{1}{n!n^n}\left|(-1)^n\begin{vmatrix} x_{11}-x_{12} & \cdots & x_{n1}-x_{n2} \\ x_{12}-x_{13} & \cdots & x_{n2}-x_{n3} \\ \vdots & \ddots & \vdots \\ x_{1n}-x_{1,n+1} & \cdots & x_{nn}-x_{n,n+1} \end{vmatrix}\right|. \tag{10}$$

1 Despite the fact that sometimes the absolute value is not explicitly stated in Lagrange's formula, it is necessary to include it, because if we exchange two rows in , the volume does not change, but the determinant changes the sign.

Finally, from (9) and (10), we obtain

$$\frac{\mathcal{V}'}{\mathcal{V}} = n^{-n}.$$

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